1.

$$z = 2 - 3\text{i}$$

(a) Show that $z^2 = -5 - 12\text{i}$.

(2)

Find, showing your working,

(b) the value of $|z^2|$,

(2)

(c) the value of $\arg(z^2)$, giving your answer in radians to 2 decimal places.

(2)

(d) Show z and z^2 on a single Argand diagram.

(1)

a) $z^2 = (2-3i)(2-3i) = 4 + 9i^2 - 12i$

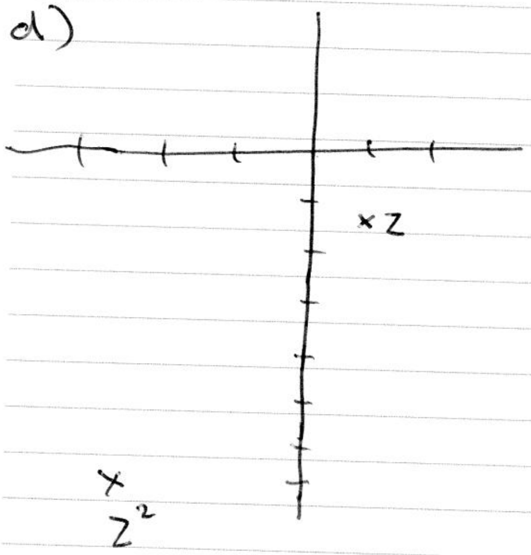
$= -5 - 12i$

b) $|z^2| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$

c) $\arg(z^2) = -\pi + \tan^{-1}\frac{12}{5} = -\pi + 1.1760052$

$= -1.97^c$

d)



2. $$\mathbf{M} = \begin{pmatrix} 2a & 3 \\ 6 & a \end{pmatrix}, \text{ where } a \text{ is a real constant.}$$

(a) Given that $a = 2$, find $\mathbf{M}^{-1}$.

(3)

(b) Find the values of a for which $\mathbf{M}$ is singular.

(2)

a) $M^{-1} = \dfrac{1}{2a^2-18} \begin{pmatrix} a & -3 \\ -6 & 2a \end{pmatrix}$

if $a=2$ $\quad M^{-1} = \dfrac{1}{-10} \begin{pmatrix} 2 & -3 \\ -6 & 4 \end{pmatrix}$

$= \begin{pmatrix} -0.2 & 0.3 \\ 0.6 & -0.4 \end{pmatrix}$

b) If M is singular, the determinant $=1$

$\therefore \; 2a^2 = 18$

$\therefore \; a = \pm 3$

3. $$f(x) = x^3 - \frac{7}{x} + 2, \quad x > 0$$

(a) Show that $f(x) = 0$ has a root α between 1.4 and 1.5

(2)

(b) Starting with the interval $[1.4, 1.5]$, use interval bisection twice to find an interval of width 0.025 that contains α.

(3)

(c) Taking 1.45 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x) = x^3 - \frac{7}{x} + 2$ to obtain a second approximation to α, giving your answer to 3 decimal places.

(5)

a) $f(1.4) = -0.256$
$f(1.5) = 0.708$
$\therefore f(x) = 0$ must lie between 1.4 and 1.5 since $f(x)$ is continuous

b) $f(1.45) = 0.221$
$\therefore \alpha$ lies between 1.4 and 1.45

$f(1.425) = -0.0186$
$\therefore \alpha$ lies between 1.425 and 1.45

c) $f'(x) = 3x^2 + \frac{7}{x^2}$

$f(1.45) = 0.221$ (above)
$f'(1.45) = 9.6368698$

$\therefore \alpha \simeq 1.45 - \frac{0.221}{9.6368698} = 1.4270672$

$= 1.427$

4. $$f(x) = x^3 + x^2 + 44x + 150$$

Given that $f(x) = (x+3)(x^2 + ax + b)$, where a and b are real constants,

(a) find the value of a and the value of b.

(2)

(b) Find the three roots of $f(x) = 0$.

(4)

(c) Find the sum of the three roots of $f(x) = 0$.

(1)

a) $3b = 150$

$\therefore b = 50$

$3ax + bx \equiv 44x$

but $b = 50$, $\therefore$ $3a + 50 = 44$

i.e. $a = -2$

b) $x^2 - 2x + 50 = 0$

$\Rightarrow x = \dfrac{2 \pm \sqrt{4 - 200}}{2} = \dfrac{2 \pm 14i}{2} = 1 \pm 7i$

$\therefore$ the three roots are $-3, 1+7i, 1-7i$

c) sum of roots $= -3 + 1 + 7i + 1 - 7i = -1$

5. The parabola C has equation $y^2 = 20x$.

(a) Verify that the point $P(5t^2, 10t)$ is a general point on C.

(1)

The point A on C has parameter $t = 4$.
The line l passes through A and also passes through the focus of C.

(b) Find the gradient of l.

(4)

a) $x = 5t^2$, $y = 10t$

$y^2 = (10t)^2 = 100t^2 = 20 \times 5t^2 = 20x$

b) The focus of C is at $\left(\frac{20}{4}, 0\right)$, i.e. $(5, 0)$

if $t = 4$, A has co-ordinates $(80, 40)$

gradient of l = $\frac{(40-0)}{80-5} = \frac{40}{75} = \frac{8}{15}$

6. Write down the 2×2 matrix that represents

(a) an enlargement with centre $(0, 0)$ and scale factor 8,

(1)

(b) a reflection in the x-axis.

(1)

Hence, or otherwise,

(c) find the matrix **T** that represents an enlargement with centre $(0, 0)$ and scale factor 8, followed by a reflection in the x-axis.

(2)

$$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix}, \text{ where } k \text{ and } c \text{ are constants.}$$

(d) Find **AB**.

(3)

Given that **AB** represents the same transformation as **T**,

(e) find the value of k and the value of c.

(2)

a) $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$

d) $AB = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix} = \begin{pmatrix} 6k+c & 0 \\ 4k+2c & -8 \end{pmatrix}$

e) $6k + c = 8$ — (1)

$4k + 2c = 0$ — (2)

(2) $\Rightarrow$ $c = -2k$ — (3)

sub (3) in (1) $6k - 2k = 8$

$\therefore$ $4k = 8$

$\therefore$ $k = 2$

$\therefore$ $c = -4$

7. $$f(n) = 2^n + 6^n$$

(a) Show that $f(k+1) = 6f(k) - 4(2^k)$. **(3)**

(b) Hence, or otherwise, prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 8. **(4)**

a) $f(k+1) = 2^{k+1} + 6^{k+1}$
$= 2 \times 2^k + 6 \times 6^k$

$6f(k) - 4(2^k) = 6 \times 2^k + 6 \times 6^k - 4 \times 2^k$
$= 6 \times 6^k + 2 \times 2^k$

$\therefore f(k+1) = 6f(k) - 4(2^k)$

b) $n = 1$ $f(1) = 2 + 6 = 8$ which is divisible by 8

Assume true for $n = k$, i.e. $f(k) = 8p$
$\therefore$ $2^k + 6^k = 8p$ where $p \in \mathbb{N}$

$f(k+1) = 6f(k) - 4(2^k)$
$= 6 \times 8p - 4 \times 2^k$
$= 8(6p) - 8(2^{k-1})$ since $k \geq 1$
$= 8(6p - 2^{k-1})$ and $6p - 2^{k-1} \in \mathbb{N}$
i.e. $f(k+1)$ is divisible by 8

$\therefore$ Proof complete by induction

8. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant.

The point A on H has x-coordinate $3c$.

(a) Write down the y-coordinate of A.

(1)

(b) Show that an equation of the normal to H at A is

$$3y = 27x - 80c$$

(5)

The normal to H at A meets H again at the point B.

(c) Find, in terms of c, the coordinates of B.

(5)

a) $\frac{1}{3}c$

b) $y = \frac{c^2}{x}$

$\therefore \frac{dy}{dx} = \frac{-c^2}{x^2}$

at $(3c, \frac{1}{3}c)$ $\quad \frac{dy}{dx} = \frac{-c^2}{9c^2} = \frac{-1}{9}$

$\therefore$ gradient of the normal is 9

$\therefore$ equation of the normal is $y - \frac{1}{3}c = 9(x - 3c)$

i.e. $3y - c = 27x - 81c$

$\therefore 3y = 27x - 80c$

c) $y = \frac{c^2}{x}$ and $y = \frac{27x - 80c}{3}$

$\therefore \frac{c^2}{x} = \frac{27x - 80c}{3}$

$\therefore 3c^2 = 27x^2 - 80xc$

Leave blank

Question 8 continued

$\therefore\ 27x^2 - 80xc - 3c^2 = 0$

$x = 3c$ is one solution

$\therefore\ (x - 3c)(27x + c) = 0$

$\therefore\ x = \frac{-c}{27} \qquad y = \frac{c^2}{x} = \frac{-27c^2}{c} = -27c$

$\therefore\ \left(-\frac{c}{27}, -27c\right)$

Leave blank

9. (a) Prove by induction that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

Using the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$,

(b) show that

$$\sum_{r=1}^{n}(r+2)(r+3) = \frac{1}{3}n(n^2+an+b),$$

where a and b are integers to be found.

(5)

(c) Hence show that

$$\sum_{r=n+1}^{2n}(r+2)(r+3) = \frac{1}{3}n(7n^2+27n+26)$$

(3)

a) $n=1$ $\quad$ LHS $=1$

RHS $= \frac{1}{6}\times 1\times 2\times 3 = 1$

$\therefore$ True for $n=1$

Assume true for $n=k$

i.e. $\sum_{1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$

Now $\sum_{1}^{k+1} r^2 = \sum_{1}^{k} r^2 + (k+1)^2$

$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right)$

$= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right)$

$= \frac{1}{6}(k+1)(2k^2+k+6k+6)$

$= \frac{1}{6}(k+1)(2k^2+7k+6)$

Leave blank

Question 9 continued

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$$

which is the required expression with n replaced by k+1

∴ proof here by induction

b) $\sum_{1}^{n}(r+2)(r+3) = \sum_{1}^{n} r^2+5r+6$

$$= \sum_{1}^{n} r^2 + 5\sum_{1}^{n} r + 6n$$

$$= \frac{n}{6}(n+1)(2n+1) + \frac{5n}{2}(n+1) + 6n$$

$$= \frac{n}{3}\left(\frac{(n+1)(2n+1)}{2} + \frac{15}{2}(n+1) + 18\right)$$

$$= \frac{n}{3}\left(\frac{2n^2+3n+1+15n+15+36}{2}\right)$$

$$= \frac{n}{3}\left(\frac{2n^2+18n+52}{2}\right)$$

$$= \frac{n}{3}(n^2+9n+26)$$

∴ $a = 9$, $b = 26$

c) $\sum_{n+1}^{2n}(r+2)(r+3) = \sum_{1}^{2n}(r+2)(r+3) - \sum_{1}^{n}(r+2)(r+3)$

$$= \frac{2n}{3}(4n^2+18n+26) - \frac{n}{3}(n^2+9n+26)$$

$$= \frac{n}{3}(8n^2+36n+52-n^2-9n-26) = \frac{n}{3}(7n^2+27n+26)$$

Q9

(Total 10 marks)

Turn over